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Appendix A from K. Cheng et al., "A Geometry of Regulatory Scaling"

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Geometric Considerations

This appendix develops the geometry used in fitting the nutritional rail data involving two dimensions of nutrients. Proofs for a number of claims are given in appendix B. To start with, different nutrients must be measured in the same units. Typically, the units will be a measure of mass (e.g., gram) or energy (e.g., kJ). The intake target in n dimensions can be characterized as $(t_1, t_2, \ldots t_n)$. For two dimensions x and y, the target point will be called (x_t, y_t).

A central enterprise of the Geometric Framework has been to determine how animals minimize discrepancies from the target point when forced onto a nutritional trajectory that cannot intersect the intake target. In the two-dimensional case, such imbalanced food rails are lines of the form $y = cx$ ($c \geq 0$ being the slope of the nutritional rail). The target point can be reached if and only if $c = y_t/x_t$. With rails for which $c \neq y_t/x_t$, we gain insight into the costs of deviations that the animal attempts to minimize.

Metrics for Nutritional Distance

Our approach takes as axiomatic that regulatory systems will tend toward minimizing discrepancies relative to a regulatory (e.g., intake) target, and it has the empirical aim of determining the relative priorities (i.e., scaling) assigned to different nutrients in circumstances where there are trade-offs in this minimization process. We take discrepancies from the target to be measured as a distance from the target. This requires us to test selected metrics, or measures of distance, in fitting data.

A powerful general set of metrics are the Minkowski metrics (Shepard 1987). Considering the two-dimensional case, which is the case in all our examples, the distance D between (x_1, y_1) and (x_2, y_2) has the form

$$D = [|(x_1 - x_2)^n| + |(y_1 - y_2)^n|]^{1/n}.$$

The most familiar metrics and the ones of relevance to biology are the cases where the exponent $n = 1$ and $n = 2$. These are the city-block and Euclidean metrics, respectively. In the city-block metric,

$$D = |(x_1 - x_2)| + |(y_1 - y_2)|. \tag{A1}$$

Deviations in the x and y dimensions are simply added. Biologically, we suppose that this metric applies if and only if the costs of deviations in each dimension increase linearly. In the Euclidean metric,

$$D = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}, \tag{A2}$$

which is a familiar formula for calculating the straight-line distance between two points on a plane. Biologically, this metric might correspond to cases in which the costs of deviations increase in an accelerating fashion, resembling a quadratic function (see "Discussion").

A further modification to the formulas (A1) and (A2) is necessary for our purposes. In the simplest scenario, a unit of deviation from the target along one dimension would be equivalent in terms of fitness consequences to the same unit of deviation in the other dimension (i.e., the costs of an excess or deficit of one nutrient would be equal to the costs of the equivalent excess or deficit of the other). However, there is no theoretical basis for the expectation that this should generally be the case; in many instances, a unit of deviation in the two dimensions would scale differently with fitness. Hence, a weighting factor for one of the dimensions (and we have arbitrarily chosen x) is needed to calibrate relative "step sizes." Deviations along x need to be weighted by a factor W, $W > 0$. If the weighting factor $W > 1$, then x is weighted more than y; if $W < 1$, then y is weighted more than x. This then leaves us with the following general formulas for distances in nutritional deviations:

$$\text{city-block metric } D = W|(x_1 - x_2)| + |(y_1 - y_2)|, \quad \text{(A3)}$$

$$\text{Euclidean metric } D = [W(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}. \quad \text{(A4)}$$

In what follows, we derive some key geometric consequences of these metrics as applied to data from nutritional rails.

City-Block Metric

By the city-block metric in equation (A3), the locus of points that minimizes D is shown in figure 1. They coincide with the line $x = x_t$ for $c < W$, with c being the slope of the nutritional rail. For $c > W$, they coincide with the line $y = y_t$. With a city-block metric then, an animal should switch abruptly from defending one nutrient to defending the other nutrient as the slope of the nutritional rail crosses W. At $c = W$, all the points on the entire segment joining the horizontal ($y = y_t$) and vertical ($x = x_t$) line segments (*gray points*, fig. 1) are equally minimum in distance from (x_t, y_t). In this range, an increased deficit/excess of one nutrient is balanced exactly by a decreased excess/deficit in the other nutrient. This lack of a unique mathematical solution is called a singularity. A variant on this pattern is the case $W = y_t/x_t$. In this case, when $c = W$, the target point (x_t, y_t) can be reached and hence comprises the unique solution. The city-block locus is just a vertical and a horizontal line segment meeting at (x_t, y_t). It remains the case, however, that when the slope of the nutritional rail crosses W, the animal switches from defending one nutrient to defending the other nutrient.

A further complication, which applies to the Euclidean metric as well, is that the weighting factor might differ for nutritional rails with slopes greater than versus less than y_t/x_t. This arises from the likely possibility that excesses and deficits of any particular nutrient might have different costs (Simpson et al. 2004). But since the weighting factor W applies to the relative costs of two nutrients, we need at most two different weighting factors. One would apply to excesses on the X-axis and deficits on the Y-axis (graphically, the bottom right quadrant with respect to the target point), and the other would apply to excesses on the Y-axis and deficits on the X-axis (the top left quadrant with respect to the target point).

For the city-block metric, the use of two weighting factors does not change the basic form of the locus of points that minimize the distance from the target. At most, an extra singularity is introduced. Most of the points should still lie parallel to the X-axis or Y-axis, intersecting the target point. When choices along nutritional rails all end up on the lines $x = x_t$ or $y = y_t$, and preferably with points on both lines, this is a signature empirical pattern for the city-block metric.

Euclidean Metric

Turning to the modified Euclidean metric in equation (A4), figure 1 shows that the locus of points defined by minimizing this metric forms part of an ellipse on an untransformed coordinate system. Using equation (A4) as the metric, however, all points are actually equidistant from ($x_t/2$, $y_t/2$), thus forming part of a circle with radius $[(Wx_t^2 + y_t^2)/4]^{1/2}$ and its center halfway between the target and the origin. Thus, in the special case of $W = 1$ or equal weighting of the two dimensions, a circle of radius $[(x_t^2 + y_t^2)/4]^{1/2}$ is described.

The weighting factor may be viewed equivalently as a transformation on the axes. Geometry can be built with axioms or with transformations. In the axiomatic approach, a set of axioms, or assumed truths, forms the starting point, and the mathematical consequences about a geometric space derived from the axioms form the geometry (Hilbert 1921). Different sets of axioms can be used to characterize different geometries. Equivalently, geometries may be defined transformationally (Klein 1939). In Klein's transformational approach, a class of geometric properties is defined as the properties that are left invariant by a set of transformations (Cheng and Gallistel 1984). Euclidean properties such as distances and angles are defined by the set of transformations consisting of translations (without rotation), rotations, and mirror reflections. For our purposes, we exclude reflections, which change a property called sense; sense is basically a left/right distinction, a property that distinguishes a right hand from a left hand (see Cheng and Gallistel 1984). Mathematically, such sets of transformations that define geometric properties are mathematical groups, but we do not consider this point, as the group structure is not relevant to the discussion at hand.

For the geometry of nutritional regulation, this means that instead of stretching a circle into an ellipse to accommodate the weighting factor W, we can equivalently transform the underlying axis system to preserve a circular locus. The transformation needed is a stretch (or shrink) along one of the axes. The *X*-axis is stretched by a factor $W^{1/2}$, a transformation known as an affine transformation (Klein 1939). As with the city-block metric, the stretch factor might differ for $x > x_t$ and $x < x_t$. But once again, at most two factors are needed, as the factor W compares x and y. One stretch factor is for $x < x_t$ and $y > y_t$; the other factor is for $x > x_t$ and $y < y_t$. This then amounts to two piece-wise affine transformations.

The Euclidean formula for distance minimizes the sums of squares of deviations, with the quadratic function capturing an accelerating cost structure. The Minkowski metrics with higher n also capture accelerating cost structures and thus might prove to be suitable metrics as well for fitting the data sets. Thus, with $n = 3$, one would attempt to minimize the sums of cubes of deviations. We have in fact fitted the nutritional data sets with Minkowski metrics with $n = 3$ and $n = 4$ (data not shown). They too provide similar fits to the Euclidean metric.

Curve-Fitting Procedure

A general aim of our procedure is to minimize the number of free parameters used in any stage of curve fitting. This makes the procedure both easier and less arbitrary. Because the curves are forced through the target, the position of the target on the graph is crucial. We estimated the target position first, and then we fixed the fitted target and proceeded with curve fitting. This is preferable to varying the target and the fitted curves simultaneously, which is more arbitrary. For example, the target position might be adjusted to favor one kind of fit versus another. Using our method, the target position is fitted "blind" to the curve fitting that follows.

Adjusting the Target

The nonhuman data sets consist of data obtained on nutritional rails and a target point with choice over two dimensions. The rail data are thus constrained to vary along a single linear trajectory (along the rail), while the target may vary over two dimensions within the bounds of the food choices offered. To reduce the arbitrary adjustment of the target point, we limited its variation to the rail on which it is found. Geometrically, this is a line connecting (0, 0) to the observed target, which can be called the target rail. We fitted the nontarget rail data with a curve chosen appropriately for each data set. The intersection of the target rail and the curve fitting the rail data defined the fitted target, which is used in curve fitting. In the human cases, we assumed the target rail to be 15% protein. The closest point to 15% protein was chosen without adjustment as the target. The fitted target point in each case was called (x_t, y_t).

Curve Fitting

For a city-block fit, the data points had to parallel the *X* and *Y* axes, at the level of the fitted target (at x_t or y_t). The weighting factors can serve to determine at what point the nutritional rails "flip" from defending one nutrient to defending the other. We never needed to choose free parameters for city-block fits, so this topic will not be discussed any further. For a Euclidean fit, the data points from rails and the fitted target were fitted first with an ellipse, with one free parameter, the weighting factor W, measuring the weight assigned to differences along the *X*-axis relative to the *Y*-axis. Errors in fits were measured as Euclidean distances along the rail on which a data point sits. (The error in fitting the fitted target need not be considered, as the target is always fitted error free.) This seemed reasonable since, as stated above, the data points could only vary along each rail rather than independently along the *Y*- or *X*-axis. Geometrically, a line is drawn from (0, 0) through the data point (the rail). The distance from the fitting ellipse to the data point along the rail defines the error. As Euclidean distances incorporate the sums of squares, this is basically a procedure of minimizing sums of squares of errors but along rails rather than along the more traditional *Y*-axis. The fitting was done by step-by-step iteration in a spreadsheet, and factor W was chosen to three decimal places.

If the fit was good and contained no systematic errors, the procedure ended. In the human case, a single ellipse resulted in systematic errors. We then fitted the data with double ellipses. One ellipse was used for protein deficits, and one was used for protein excesses, relative to the chosen target. The chosen target was included in both ellipses, but again, it was irrelevant in curve fitting, as the ellipses were constrained to go through the target.

Refinements Needed Arising from Partial Substitutability of Nutrients

One biological factor that can change the geometric shape of data nutrient intake array is the partial substitutability of different nutrients. For example, generalist feeders such as the locust *Schistocerca gregaria* possess better abilities to convert between proteins and carbohydrates than specialist feeders such as the locust *Locusta migratoria* (Raubenheimer and Simpson 2003). Physiologically, proteins may be deaminated, or stripped of nitrogen, to obtain carbohydrates. In the other direction, while proteins cannot be created out of carbohydrates (which lack nitrogen), nitrogen may be better conserved in the consumed protein to cope with a lower-than-target level of protein consumption. The two directions of partial intersubstitutability need not be equal. It is likely, for example, that the protein-to-carbohydrate substitutability is higher than the substitutability in the reverse direction.

Partial substitutability produces a characteristic geometric signature. In the limit, when two nutrients are completely substitutable for one another, the locus of points on nutritional rails should form a straight line through the target point with a slope of -1. This is true of both the city-block and Euclidean metrics. On this line, the total amount of nutrients consumed remains constant. Geometrically, partial substitutability drives the points toward this line. The degree of substitutability can be characterized as a proportion (between 0 and 1). Thus, a substitutability proportion of 0.5 means that half of the excess in one nutrient can serve to substitute for a deficit in the second nutrient. A substitutability proportion greater than 0.5 makes the locus of points on nutritional rails approach a straight line of slope -1.

The "translation" of partial substitutability to geometric distance is straightforward for any metric. Consider for example the case of rails with an excess of x and a deficit of y; these are rails $y = cx$, with $c < y_t/x_t$. Take a point (x, y) that exceeds x_t on the X-axis but falls short of y_t on the Y-axis ($x > x_t$ and $y < y_t$). Suppose that $W = 1$, an assumption that we will make, meaning that deviations in the two dimensions are equally weighted. Normally, distance from the target along the X-axis is $x - x_t$. Partial substitutability means that this distance is reduced by a proportion b ($0 \leq b \leq 1$) of the excess on the Y-axis: $x - x_t - b(y_t - y)$. Following the same logic, the distance from the target along the Y-axis would be: $y_t - y - b(x - x_t)$. An analogous process, with the same b, is used to adjust the distances from the target along the two axes for deficits in x and excesses in y.

After these adjustments, the standard formulas for distance (Euclidean, city-block, or any other) can be applied. In our curve fitting, we set $W = 1$, to eliminate a free parameter, and adjusted b for the best fit. Using Euclidean distance in such curve fits, increasing b "unbends" a circle to the limit of a straight line of slope -1 through the target point. The distance-minimizing points are "driven" quickly to this line with increasing b. Even with a b of 0.5, the points that minimize the distance to the target approach this line with a slope of -1. We found that a small b of 0.069 improves the fit to the data from *Locusta* (protein vs. carbohydrates; fig. 5) but that the best-fitting function using the Euclidean metric for *Schistocerca* (fig. 6*B*; table 1) has a sizeable b of 0.760.

Using the city-block metric, increasing b increases the angle formed by two straight segments comprising the distance-minimizing points. Setting $W = 1$ again, at $b = 0$, the locus of points at minimal distance from the target parallel the axes and intersect at the target. The segment parallel to the X-axis is at the target value for nutrient y, while the segment parallel to the Y-axis is at the target value for nutrient x. Increasing b widens the angle at which the two line segments intersect (still at the target point). The slope of the segment with deficits of x is $-b$, while the slope of the segment with excesses of x is $-1/b$. In a sense, b represents a trade-off ratio that is reflected in the slope. Thus, at $b = 1$, the two segments form a straight line of slope -1 running through the target point. This technique does not fit the *Locusta* data in figure 5 well because the data set forms a curve and not two straight lines. But it provides an excellent fit for the data from *Schistocerca* in figure 6. If anything, the city-block fit was slightly better than the Euclidean fit, with a best-fitting residual error of 6.233, measured as city-block distance. The best fitting value for b, however, was 1. This would imply indifference between the two nutrients and predict that the locusts would not regulate to a single target point, contrary to the empirical evidence.

Operant Data

In "Discussion," we fitted two sets of data from operant conditioning. In operant data, two dimensions of behaviors are scaled. One is the operant, what the animal has to do to get enough of the other behavior, the reinforcer (reward). Equilibrium theory says that there are ideal levels of each behavior that the animal prefers to do. Because the target level of the operant response is very low, both data sets shown in figure 8, and in fact most operant data, have rails in which only the operant behavior is in excess. But both data sets contain steep enough rails (lean enough operant schedules) that the left side of the ellipse (with positive slope) is evident. In figure 8*A*, the target point was taken from Hanson and Timberlake's (1983) figure 3, in which they gave the values of the target. The data were fitted by our standard procedures, without target adjustment. In figure 8*B*, a fitted target was first calculated as the intersection of the target rail and a quadratic function fitting the rail data. We then fitted the data by eye using a single ellipse instead of any formal minimization procedure. That is because for steep rails that intersect the left side of the ellipse, errors as measured along rails can be very large. Fitting by minimizing errors along rails is then dominated by fitting the leftmost points at the expense of fitting other points. Figure 8 shows that an elliptical fit is reasonable in some cases of operant data.

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Derivations and Proofs

Minimizing City-Block Distance to a Target Point Along a Nutritional Rail

In the Geometric Framework, the intake target has units of consumption over a set period of time (e.g., 1 d). The same units apply to both axes, for example, both in milligrams or kilojoules. The intake target is set over two nutritional dimensions, x and y, at $(\mathrm{x_t}, \mathrm{y_t})$. We use italics for variables and normal font for constants.

A nutritional rail is the locus of points satisfying the equation $y = \mathrm{c}x$, for $\mathrm{c} \geq 0$, plus the rail $x = 0$. Call the city-block distance of $(x, \mathrm{c}x)$ to $(\mathrm{x_t}, \mathrm{y_t})$ $|D|$:

$$|D| = |x - \mathrm{x_t}| + |y - \mathrm{y_t}|.$$

We suppose that the x and y dimensions are in general weighted differently. This amounts to adding a weighting factor W ($\mathrm{W} > 0$) to the x part of the distance calculation:

$$|Dw| = \mathrm{W}|x - \mathrm{x_t}| + |\mathrm{c}x - \mathrm{y_t}|.$$

The solution needs to be divided case by case.

Case 1: $\mathrm{c} < \mathrm{W}$

Minimum distance is at $(\mathrm{x_t}, \mathrm{c} \times \mathrm{x_t})$: $|Dw| = \mathrm{W}|\mathrm{x_t} - \mathrm{x_t}| + |\mathrm{c} \times \mathrm{x_t} - \mathrm{y_t}| = |\mathrm{c} \times \mathrm{x_t} - \mathrm{y_t}|$.

Proof. If $x > \mathrm{x_t}$, then $x = \mathrm{x_t} + \mathrm{e}$, $\mathrm{e} > 0$.

$$|Dw| = \mathrm{W}|(\mathrm{x_t} + \mathrm{e}) - \mathrm{x_t}| + |\mathrm{c}(\mathrm{x_t} + \mathrm{e}) - \mathrm{y_t}|$$
$$= \mathrm{W} \times \mathrm{e} + |\mathrm{c} \times \mathrm{x_t} + \mathrm{ce} - \mathrm{y_t}|.$$

This expression exceeds $|\mathrm{c} \times \mathrm{x_t} - \mathrm{y_t}|$ since $\mathrm{W} > \mathrm{c}$, and hence $\mathrm{W} \times \mathrm{e} > \mathrm{c} \times \mathrm{e}$. Distance $|Dw|$ might be reduced by $\mathrm{c} \times \mathrm{e}$, but it increases by $\mathrm{W} \times \mathrm{e}$.

If $x < \mathrm{x_t}$, then $x = \mathrm{x_t} - \mathrm{e}$, $\mathrm{e} > 0$, with e being some arbitrary increment to $\mathrm{x_t}$.

$$|Dw| = \mathrm{W}|(\mathrm{x_t} - \mathrm{e}) - \mathrm{x_t}| + |\mathrm{c}(\mathrm{x_t} - \mathrm{e}) - \mathrm{y_t}|$$
$$= \mathrm{W}|-\mathrm{e}| + |\mathrm{c} \times \mathrm{x_t} - \mathrm{ce} - \mathrm{y_t}|.$$

This expression also exceeds $|\mathrm{c} \times \mathrm{x_t} - \mathrm{y_t}|$ since $\mathrm{W} > \mathrm{c}$, and hence $\mathrm{W} \times \mathrm{e} > \mathrm{c} \times \mathrm{e}$. Distance $|Dw|$ might be reduced by $\mathrm{c} \times \mathrm{e}$, but it increases by $\mathrm{W} \times \mathrm{e}$.

Case 2: $\mathrm{c} > \mathrm{W}$

Minimum distance is at $(\mathrm{y_t}/\mathrm{c}, \mathrm{y_t})$: $|Dw| = \mathrm{W}|\mathrm{y_t}/\mathrm{c} - \mathrm{x_t}| + |\mathrm{y_t} - \mathrm{y_t}| = \mathrm{W}|\mathrm{y_t}/\mathrm{c} - \mathrm{x_t}|$.

Proof. If $x > \mathrm{y_t}/\mathrm{c}$, then $x = \mathrm{y_t}/\mathrm{c} + \mathrm{e}$, $\mathrm{e} > 0$, and $y = \mathrm{y_t} + \mathrm{ce}$.

$$|Dw| = W|x_t + e - x_t| + |c \times x_t - y_t + c \times e|$$
$$= W \times e + |c \times x_t - y_t + ce|.$$

Given that $W = c > 0$, this expression exceeds $|c \times x_t - y_t| = |y_t - W \times x_t|$ by $2 \times W \times e$.
If $x < y_t/c$, then $x = y_t/c - e$, $e > 0$, and $y = y_t - ce$.

$$|Dw| = W\left(\left| y_t/c - x_t - e \right|\right) + |y_t - ce - y_t|$$
$$= |y_t - W \times x_t - W \times e| + |-ce|.$$

Again, given that $W = c > 0$, this expression exceeds $|c \times x_t - y_t| = |y_t - W \times x_t|$ by $2 \times W \times e$.

Formula for Minimizing Euclidean Distance to a Target Point along a Nutritional Rail

Again, the target over two nutritional dimensions x and y is at (x_t, y_t), and a nutritional rail is the locus of points satisfying the equation $y = cx$, for $c \geq 0$. This leaves out the rail $x = 0$, which can be solved by taking the limit as $c \to \infty$. The line $y = cx$ is the locus of points (x, cx), for $c \geq 0$. Call the Euclidean distance of (x, cx) to (x_t, y_t) D:

$$D = \sqrt{(x - x_t)^2 + (cx - y_t)^2}.$$

Again, we suppose that the x and y dimensions are in general weighted differently. This amounts to adding a weighting factor W ($W > 0$) to the x part of the distance calculation:

$$Dw = \sqrt{W(x - x_t)^2 + (cx - y_t)^2}.$$

To find the minimum of Dw, it is easier to find the x that minimizes $(Dw)^2$.

$$(Dw)^2 = W(x - x_t)^2 + (cx - y_t)^2$$
$$= W \times x^2 - 2 \times W \times x_t \times x + W \times x_t^2 + c^2x^2 - 2c \times y_t \times x + y_t^2$$
$$= (c^2 + W)x^2 - 2(W \times x_t + c \times y_t)x + W \times x_t^2 + y_t^2.$$

At the minimum, the slope is 0. To find the minimum of this quadratic equation, we take the derivative of $(Dw)^2$ and solve for the equation that sets it to 0.

$$(Dw(x, y)^2)' = 2(c^2 + W)x - 2(W \times x_t + c \times y_t).$$

Setting this expression to 0 gives

$$0 = 2(c^2 + W)x - 2(W \times x_t + c \times y_t).$$

Solving for x delivers

$$x = \frac{W \times x_t + c \times y_t}{W + c^2},$$

$$y = cx = \frac{c \times W \times x_t + c^2 \times y_t}{W + c^2}.$$

The limit of this locus of points as $c \to \infty$ is $(0, y_t)$. For x, as c gets big, c^2 gets much bigger compared to c, and that makes $x \to 0$. For y, both $c \times W$ and W become insignificant compared to c^2 as c gets big, and that makes $y \to c^2 \times y_t/c^2 = y_t$.

Proof That These Points Define a Circle by the *Dw* Metric

This locus of points,

$$x = \frac{W \times x_t + c \times y_t}{W + c^2},$$

$$y = \frac{c \times W \times x_t + c^2 \times y_t}{W + c^2},\ c > 0$$

defines a circle around $(0.5 \times x_t, 0.5 \times y_t)$ by the *Dw* metric, with radius $[(W \times x_t^2 + y_t^2)/4]^{1/2}$. To prove this, it is easier to show that

$$Dw(x, y)^2 = \frac{W \times x_t^2 + y_t^2}{4},$$

$$Dw(x, y)^2 = W\left(x - \frac{x_t}{2}\right)^2 + \left(y - \frac{y_t}{2}\right)^2,$$

$$= W\left(\frac{Wx_t + c \times y_t}{W + c^2} - \frac{x_t^2}{2}\right) + \left(\frac{cWx_t + c^2y_t}{W + c^2} - \frac{y_t^2}{2}\right).$$

Collecting common denominators,

$$Dw(x, y)^2 = W\left(\frac{Wx_t + 2c \times y_t - c^2 \times x_t}{2c^2 + 2W}\right)^2 + \left(\frac{2cWx_t + c^2y_t - Wy_t}{2c^2 + 2W}\right)^2.$$

Squaring the expressions,

$$Dw(x, y)^2 = \frac{W(W^2x_t^2 + 2Wx_ty_tc - Wx_t^2c^2 + 2Wx_ty_tc + 4y_t^2c^2 - 2x_ty_tc^3 - Wx_t^2c^2 - 2x_ty_tc^3 + x_t^2c^4)}{4c^4 + 8Wc^2 + 4W^2} + \frac{4W^2x_t^2c^2 + 2Wx_ty_tc^3 - 2W^2x_ty_tc + 2Wx_ty_tc^3 + y_t^2c^4 - Wy_t^2c^2 - 2W^2x_ty_tc - Wy_t^2c^2 + W^2y_t^2}{4c^4 + 8Wc^2 + 4W^2}$$

Collecting terms and organizing by powers of c,

$$Dw(x, y)^2 = \frac{W[x_t^2c^4 - 4x_ty_tc^3 + (4y_t^2 - 2x_t^2W)c^2 + 4Wx_ty_tc + W^2x_t^2)] + [y_t^2c^4 + 4Wx_ty_tc^3 + (4W^2x_t^2 - 2Wy_t^2)c^2 - 4W^2x_ty_tc + W^2y_t^2]}{4(c^4 + 2Wc^2 + W^2)}$$

$$= \frac{Wx_t^2c^4 - 4Wx_ty_tc^3 + 4Wy_t^2c^2 - 2W^2x_t^2c^2 + 4W^2x_ty_tc + W^3x_t^2 + y_t^2c^4 + 4Wx_ty_tc^3 + 4W^2x_t^2c^2 - 2Wy_t^2c^2 - 4W^2x_ty_tc + W^2y_t^2}{4(c^4 + 2Wc^2 + W^2)}.$$

The c^3 and c terms drop out, leaving

$$
\begin{aligned}
Dw(x, y)^2 &= \frac{(\mathrm{W}\mathrm{x}_t^2 + \mathrm{y}_t^2)\mathrm{c}^4 + 2\mathrm{W}\mathrm{y}_t^2\mathrm{c}^2 + 2\mathrm{W}^2\mathrm{x}_t^2\mathrm{c}^2 + \mathrm{W}^3\mathrm{x}_t^2 + \mathrm{W}^2\mathrm{y}_t^2}{4(\mathrm{c}^4 + 2\mathrm{a}\mathrm{c}^2 + \mathrm{a}^2)} \\
&= \frac{(\mathrm{W}\mathrm{x}_t^2 + \mathrm{y}_t^2)\mathrm{c}^4 + (\mathrm{W}\mathrm{x}_t^2 + \mathrm{y}_t^2)2\mathrm{W}\mathrm{c}^2 + (\mathrm{W}\mathrm{x}_t^2 + \mathrm{y}_t^2)\mathrm{a}^2}{4(\mathrm{c}^4 + 2\mathrm{W}\mathrm{c}^2 + \mathrm{W}^2)} \\
&= \frac{(\mathrm{W}\mathrm{x}_t^2 + \mathrm{y}_t^2) \times (\mathrm{c}^4 + 2\mathrm{W}\mathrm{c}^2 + \mathrm{W}^2)}{4(\mathrm{c}^4 + 2\mathrm{W}\mathrm{c}^2 + \mathrm{W}^2)} \\
&= \frac{\mathrm{W}\mathrm{x}_t^2 + \mathrm{y}_t^2}{4}.
\end{aligned}
$$